① Finish Gauss lemma

② Hopf - Rinow

1] Recall

Thm Geodesics are locally length-minimizing.

Fix $\mathbb{R}^n \simeq T_pM$, let g_{ij} be pull-back metric,

$\quad\;\; \mathcal{U} \xrightarrow{\exp_p} M$

$r = \sqrt{\delta_{ij} x^i x^j}$ $\qquad dr = \frac{x^i dx^i}{r}$

Gauss's lemma: $g(dr, dr) = 1$

Recall $g^{ij} = g(dx^i, dx^j)$

$g(dr, dr) = \frac{x^i x^j}{r^2} g^{ij}$

Claimed If $\partial_r = \frac{x^i}{r}\frac{\partial}{\partial x^i}$, suffices to show
- $g(\partial_r, \partial_r) = 1$
- $\ker(dr) = \ker(g(\partial_r, \cdot))$

i.e. $g(\partial_r, \cdot) = dr(\cdot)$ i.e. $\partial_r = \text{grad } r$

and $|\partial_r|_g = 1$

and $|\text{grad } r| = |dr|$

Pf 1) $\partial_r g(\partial_r, \partial_r) = 0$, $\lim_{r\to 0} g(\partial_r, \partial_r) = 1$

(2) If ∂_θ tangent vector to S^n, $\partial_\theta = A^i_j x^j \frac{\partial}{\partial x^i}$ for some skew-sym matrix A_{ij}

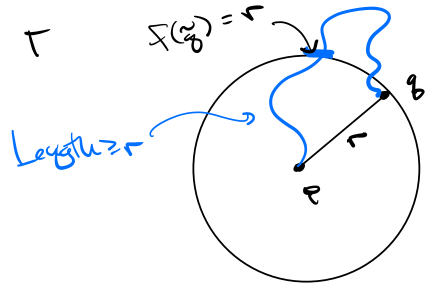
$g(\partial_r, \partial_\theta) = 0$

$\partial_r g(\partial_r, \partial_\theta) = \frac{1}{2}\partial_\theta g(\partial_r, \partial_r) = 0$

$\lim_{r\to 0} g(\partial_r, \partial_\theta) = \frac{x^i}{r^2} A^j_k x^k (\delta_{ij} + o(1)) = 0$ ✓

Cor In $\mathcal{U}$, $r(q) = d(p,q)$

Pf



$\Rightarrow$ any path from p to q has length $\geq r$ ⌋

Cor Geodesics are locally minimizing

Pf If γ is a geod, $p \in \gamma$, let "$\mathcal{U} \times B_\varepsilon$" $\subseteq TM$

be a nbhd s.t. $\exp_q|_{B_\varepsilon}$ is a diffeo $\forall\, q \in \mathcal{U}$. (IFT)

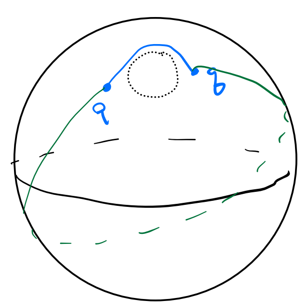
then we can apply this arg for any $q \in \mathcal{U}$ ⌋

Rmk: This gives another proof of:

Every conn Riem mfld is a metric space.

Q: Is $d(p,q)$ always realized by a geodesic?

A: No, e.g.



Defn S is <u>geodesically complete</u> if every (parametrized) geodesic segment $\alpha : (0, \varepsilon) \to S$ is part of a geodsic $\bar{\alpha} : \mathbb{R} \to S$ that exists for all time, i.e. if $\exp_p(w)$ is defined $\forall\, p, w$.

Prop A closed surface is geodesically complete.

(compare analogous thm for flows)

Γ Suppose S is closed, and let $\alpha : (0, T) \to S$ be a geodesic segment, parametrized at unit speed.

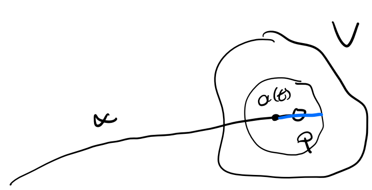
Since α is unit speed, the path $\alpha(t) \in \mathbb{R}^3$ has a limit point $p \in \mathbb{R}^3$ as $t \to T$; since S is closed, $p \in S$.

$t_i \to T$
$\{\alpha(t_i)\}$ is Cauchy $\rightsquigarrow$ has a limit which doesn't depend on $\{t_i\}$.

Let V be a uniformly normal neighborhood of p, such that $\exp_q$ is defined on $B_\varepsilon(0) \in T_q S$ $\forall\, q \in V$, and choose t close enough to T that

- $\alpha(t) \in V$
- $t > T - \frac{\varepsilon}{2}$

(b/c convergence in $\mathbb{R}^3$ to = converge in S to p)



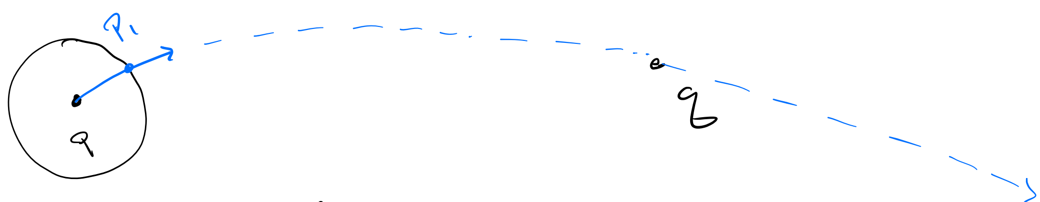
Then the geodesic segment through $\alpha(t)$ with derivative $\alpha'(t)$ extends α to time $T+\frac{\varepsilon}{2}$.

To complete the proof, let $T = \sup\{t\in\mathbb{R} \mid \alpha \text{ extends to } (0,t)\}$

If $T<\infty$, the above argument gives a contradiction.

<u>Thm</u> If S is geodesically complete, then any pair of points is connected by a distance-minimizing geodesic, i.e. a geodesic of length $d(p,q)$.

Pf Given $p,q\in S$, let $r=d(p,q)$. To find the best direction to set off from p, let ε be such that $\exp_p$ is injective on $\overline{B_\varepsilon}$, and let $p_1\in\exp_p(\partial B_\varepsilon)$ be a point for which $d(p_1,q)$ is minimized (compared to other points in $\exp_p(\partial B_\varepsilon)$).



Let α be the unit speed geodesic from p to p_1. We show

$$d(\alpha(t),q) = r-t \quad \forall \quad t\in[0,r]$$

Pf
- it is true for $t=0$ b/c $\alpha(0)=p$
- it is true for $t=\varepsilon$ b/c every path from p to q meets $\exp(\partial B_\varepsilon)$, so

$$r=d(p,q) = \inf\left(d(p,p')+d(p',q)\right)$$

$$\qquad\qquad {}_{q'\in\exp(\partial B_\varepsilon)}$$

$$= \varepsilon + \inf_{q'\in\exp(\partial B_\varepsilon)} d(q',q)$$

$$= \varepsilon + d(p_1,q)$$

- it is true for $t \le \varepsilon$ b/c for $t \le \varepsilon$,

$$d(\alpha(t),q) \le d(\alpha(t),p_1) + d(p_1,q) \qquad (\Delta \text{ ineq})$$

$$\le (\varepsilon - t) + (r - \varepsilon)$$

$$= r - t$$

but $r = d(\alpha(0),q) \le d(\alpha(0),\alpha(t)) + d(\alpha(t),q)$

$$\le t + d(\alpha(t),q)$$

$\Rightarrow \quad r - t \le d(\alpha(t),q)$.

- If T is the largest time for which it is true and $T < r$, we can do the same trick with $q = \alpha(T)$ ⌟

Hence $d(\alpha(r),q) = 0$, so $\alpha(r) = q$. ⌟

Thm (Hopf-Rinow) S is geodesically complete iff and only if the metric space (S,d) is complete

① suppose geod complete, let p_n be a cauchy seq. let $v_n \in T_pM$ be s.t. $p_n = \exp(v_n)$ and $|v_n| = d(p,p_n)$. then $v_n \xrightarrow[\text{sub}]{} v$, and by continuity of exp, $\exp(v) = \lim_{n_i} \exp(v_{n_i})$. cauchy $\Rightarrow p_n \to \exp(v)$.

(D) Suppose metrically complete, not geod complete.

let $\gamma : [0,b) \to M$ maximal geodesic (unit speed)

$\gamma(t_i)$ Cauchy as $t_i \to b$, let $q = \lim \gamma(t_i)$

U u.n.n of q, use to extend geodesic. $\lrcorner$